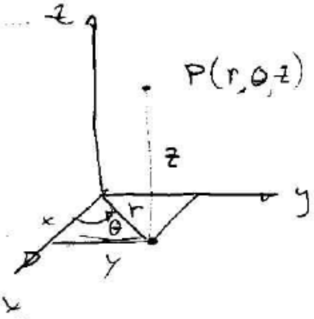
§10.7 Cylindrical + Spherical Coord

Cylindrical coord:



Can specify $P(x,y,z)$
or $P(r,\theta,z)$

From geom $z = z$
$x^2+y^2 = r^2$

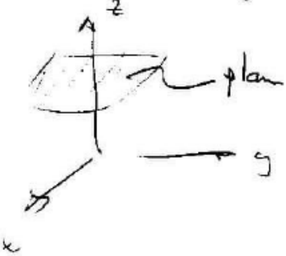
$x = r\cos\theta$
$y = r\sin\theta$

so $\frac{y}{x} = \tan\theta$

$$\left\{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{array}\right\}$$

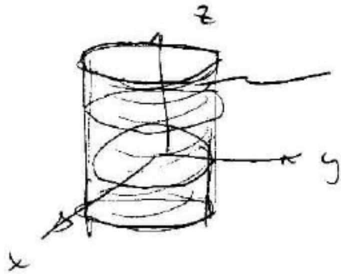
Iso-surfaces in cylindrical coord.

z = ¢



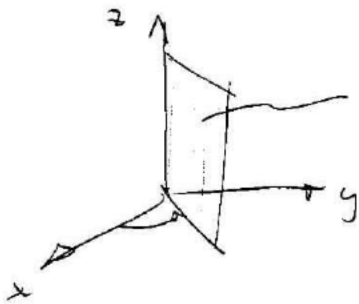
plane with z = ¢
but r, θ vary

r = ¢



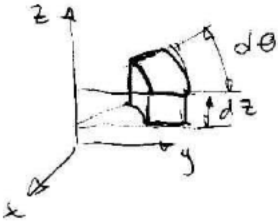
circular cylinder with r = ¢
but θ and z vary.

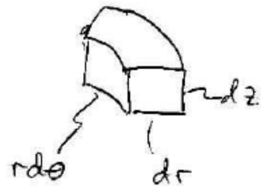
θ = ¢



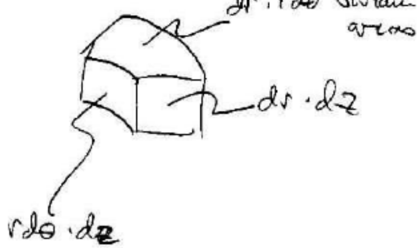
plane θ = ¢
but r, z vary

Elemental volumes and surfaces



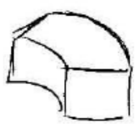


edge lengths



volume

$dV = r d\theta \cdot dr \cdot dz$



or

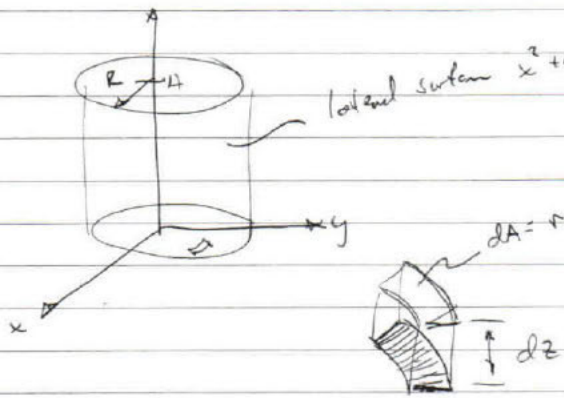
 $dA \cdot l = dr dz \cdot r d\theta$

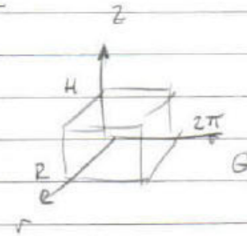
 $dA \cdot l = r d\theta dz \cdot dr$

 $dA \cdot l = dr\, r d\theta \cdot dz$

all give $dV = r d\theta\, dr\, dz$

Ex





$$V = \int\int\int 1 \cdot \underbrace{r\,dr\,d\theta\,dz}_{dV}$$

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{R}\int_{z=0}^{H} r\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R} rH\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{R^2}{2}H\,d\theta = 2\pi\left(\frac{R^2}{2}H\right) = \pi R^2 H$$

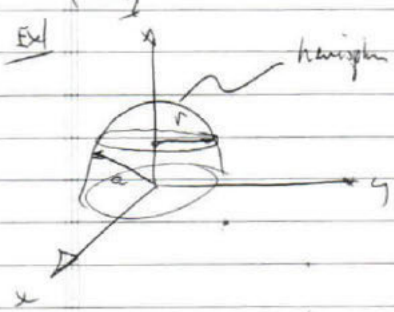
New order r, θ, z

13.6

2

$\boxed{\text{Cylindrical coord}}$

Ex

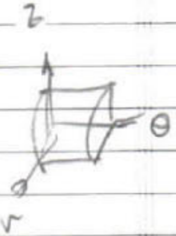


hemisphere $\quad x^2+y^2+z^2=a^2$

$$r^2+z^2=a^2$$

$$r=+\sqrt{a^2-z^2}$$

$$V=\int\int\int (r\,d\theta)(dz)(dr)$$



$$=\int_{\theta=0}^{2\pi}\int_{z=0}^{z=a}\int_{r=0}^{r=\sqrt{a^2-z^2}} r\,dr\,dz\,d\theta$$

$$\frac{r^2}{2}\Big|_0^{\sqrt{a^2-z^2}}=\frac{a^2-z^2}{2}$$

$$=\int_{\theta=0}^{2\pi}\int_{z=0}^{a}\left(\frac{a^2-z^2}{2}\right)dz\,d\theta$$

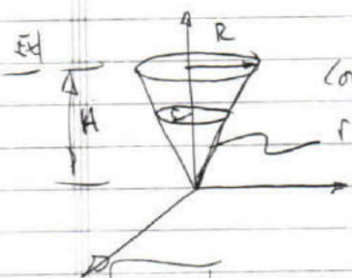
$$\frac{a^2}{2}z-\frac{z^3}{6}$$

$$\frac{a^3}{2}-\frac{a^3}{6}=\frac{2a^3}{6}$$

$$=\int_{\theta=0}^{2\pi}\frac{2a^3}{6}\,d\theta=\frac{1}{3}a^3\cdot 2\pi$$

$$=\frac{2\pi}{3}a^3$$

Ex



$V = \frac{\pi}{3} R^2 H$

Cone

$r = \left(\frac{R}{H}\right) z$

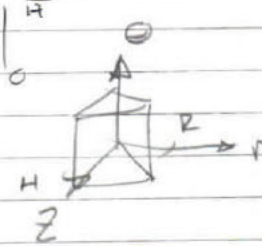
or $z = \left(\frac{H}{R}\right) r$

use dr, dz, $d\theta$

$\boxed{\text{Cyl.}}$ $\quad r = \frac{R}{H} z$

$$V = \int_{\theta=0}^{2\pi} \int_{z=0}^{H} \int_{r=0}^{\left(\frac{R}{H}\right) z} r \, dr \, dz \, d\theta$$

$$\left. \frac{r^2}{2} \right|_0^{\frac{Rz}{H}}$$

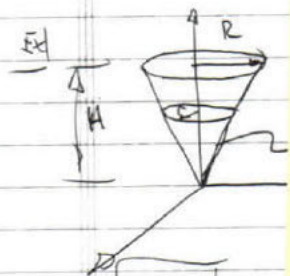


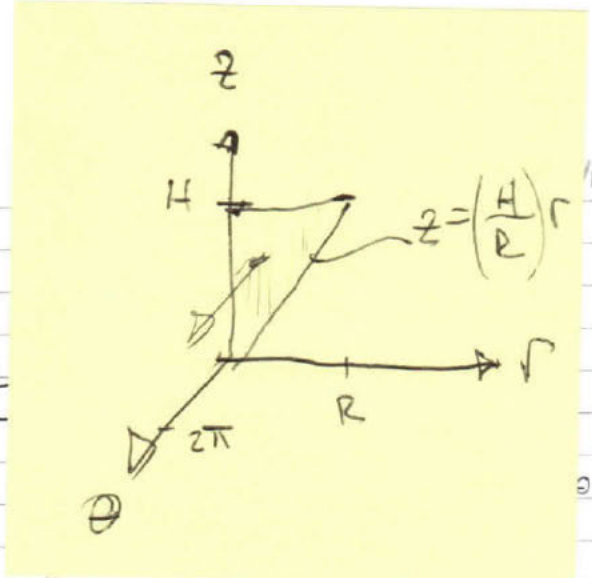
$$= \int_{\theta=0}^{2\pi} \int_{z=0}^{H} \frac{R^2}{H^2} \frac{z^2}{2} \, dz \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{R^2}{H^2} \frac{H^3}{6} \, d\theta$$

$$= \frac{R^2}{6} H \, 2\pi = \frac{\pi}{3} R^2 H$$

Ex



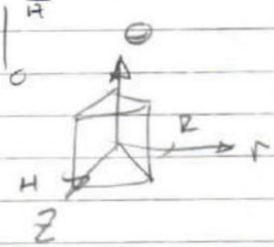


$\boxed{\text{Cyl.}}$

$$V = \int_{\theta=0}^{2\pi} \int_{z=0}^{H} \int_{r=0}^{\left(\frac{R}{H}\right)z} r \, dr \, dz \, d\theta$$

$$\frac{r^2}{2}\Big|_0^{\frac{Rz}{H}}$$

$$= \int_{\theta=0}^{2\pi} \int_{z=0}^{H} \frac{R^2}{H^2} \frac{z^2}{2} \, dz \, d\theta$$



$$= \int_{\theta=0}^{2\pi} \frac{R^2}{H^2} \frac{H^3}{6} \, d\theta$$

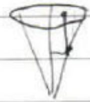
$$= \frac{R^2}{6} H \, 2\pi = \frac{\pi}{3} R^2 H$$

13.6

$\boxed{\text{Cylindrical}}$

4 /4

Same, but use $dz,\ dr\ d\theta$



$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \int_{z=\left(\frac{H}{R}\right)r}^{H} r \, dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r\left(H - \frac{H}{R} r\right) dr \, d\theta$$

$$\left(\frac{H r^2}{2} - \frac{H}{R} \frac{r^3}{3} \right|_0^R$$

$$HR^2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$V = \int_{\theta=0}^{2\pi} \frac{HR^2}{6} \, d\theta$$

$$= \frac{2\pi}{6} HR^2$$

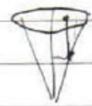
$$V = \frac{\pi}{3} HR^2$$

13.6

$\boxed{\text{Cylindrical}}$

4 /4

Same, but use $dz,\ dr\ d\theta$



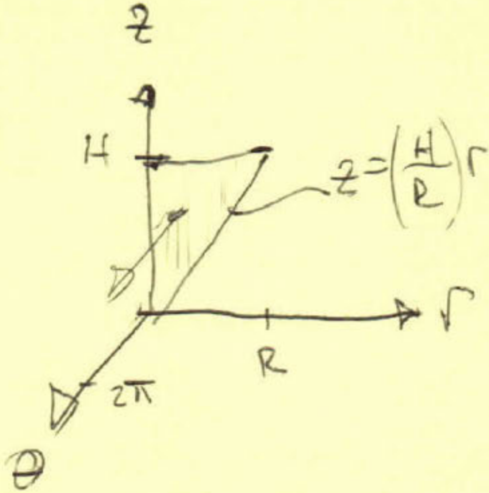
$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \int_{z=\left(\frac{H}{R}\right)r}^{H} r\, dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r\left(H - \frac{H}{R} r\right) dr\, d\theta$$

$$V = \int_{\theta=0}^{2\pi} \frac{H}{6}$$

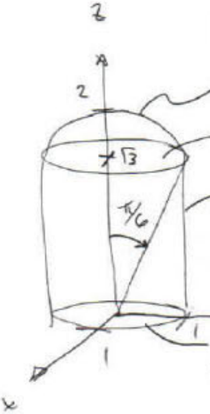
$$= \frac{2\pi}{6} H$$

$$V = \frac{\pi}{3} H$$

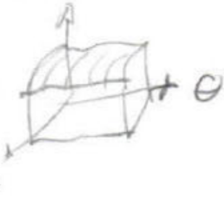


§ 13.6 cont Volumes in Cyl + Sph Coord

#11



Find volume V of can with hemi-spher top



In cylindrical coord.

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} 1 \cdot r \underline{(dz)} \underline{(dr)} \underline{(d\theta)}$$

or

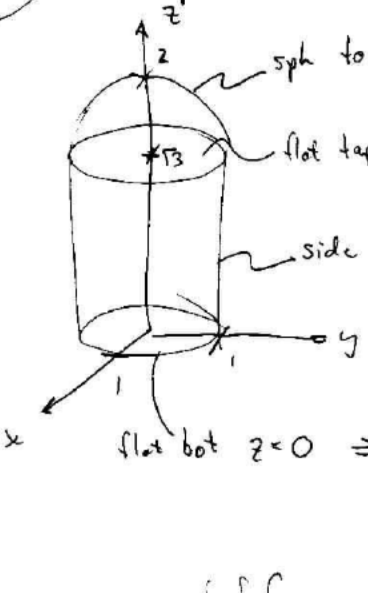
$$V = \int_{\theta=0}^{2\pi} \int_{z=0}^{\sqrt{3}} \int_{r=0}^{r=1} 1 \cdot r \underline{(dr)} \underline{(dz)} \underline{(d\theta)} \quad \text{can}$$

$$+ \int_{\theta=0}^{2\pi} \int_{z=\sqrt{3}}^{2} \int_{r=0}^{r=\sqrt{4-z^2}} 1 \cdot r (dr)(dz)(d\theta) \quad \text{dome on can}$$

#11 or

$$V = \int_{r=0}^{1} \int_{z=0}^{z=\sqrt{4-r^2}} \int_{\theta=0}^{2\pi} 1\cdot r\,(d\theta)(dz)\,dr$$

#31 Same problem but in spherical coord.



sph top $x^2+y^2+z^2=2^2 \Rightarrow \boxed{\rho^2 = 2^2}$ top

flat top $z=\sqrt{3} \Rightarrow \boxed{\rho\cos\varphi = \sqrt{3}}$

side $x^2+y^2=1 \Rightarrow r^2 = 1^2$

$\Rightarrow (\rho\sin\varphi)^2 = 1^2$

$\Rightarrow \boxed{\rho\sin\varphi = 1}$ side

flat bot $z=0 \Rightarrow \boxed{\rho\cos\varphi = 0} \Rightarrow \varphi = \pi/2$ bottom

$$V = \iiint 1\cdot \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \iiint 1\cdot \underbrace{(d\rho)(\rho\, d\varphi)(\rho\sin\varphi\, d\theta)}_{dV}$$